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Weakly coupled oscillators and topological degree

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Abstract

By using the topological degree of Brouwer for mappings along with averaging method, we study the existence of forced periodic solutions for certain weakly coupled periodically perturbed ordinary differential equations.

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1. Introduction

The purpose of this paper is to extend a method developed in [2] for more general differential systems. We consider a coupled system of two ordinary differential equations when the first equation is weakly nonlinear with respect to a small parameter ε while the second equation is strongly nonlinear. We suppose in addition that the perturbation of the system is 1-periodic in the

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time. So we are interested in forced 1-periodic solutions of the system. By using a combination of the Liapunov–Schmidt method together with the Brouwer degree theory, we find conditions for the existence of 1-periodic solutions. We consider two cases for the second equation of the unperturbed autonomous system with $\varepsilon = 0$ and with fixed but arbitrary first variable: the first case is when it has a single 1-periodic solution and the second case is when it has a nondegenerate family of 1-periodic solutions. We derive the first order bifurcation functions for the both cases. The higher order ones could be also derived, but since the computations are rather awkward we omit them. We end the paper with presenting two illustrative examples. The averaging method is also used, for instance, in the papers [1,3,5,6].

2. Forced oscillations

We start with the following equation

$$\begin{aligned} x' &= \varepsilon f(x, y, t, \varepsilon), \\ y' &= g(x, y) + \varepsilon h(x, y, t, \varepsilon), \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, f, g, h are sufficiently smooth, f, h are 1-periodic in t and ε is a small parameter. We suppose the condition

(H1) $y' = g(x, y)$ has a 1-periodic smooth solution $y = \varphi(t, x)$ for any $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open subset.

We are looking for 1-periodic solutions of (2.1) bifurcating from $\varphi(t, x)$, $x \in \Omega$. To this end, we shift $t \leftrightarrow t + \alpha$, $\alpha \in \mathbb{R}$ and change $y \leftrightarrow y + \varphi(t, x)$ to get the modified equation

$$\begin{aligned} x' &= \varepsilon f\big(x, y + \varphi(t, x), t + \alpha, \varepsilon\big), \\ y' &= g\big(x, y + \varphi(t, x)\big) - g\big(x, \varphi(t, x)\big) \\ &\quad + \varepsilon\Big(h\big(x, y + \varphi(t, x), t + \alpha, \varepsilon\big) - \varphi_x(t, x) f\big(x, y + \varphi(t, x), t + \alpha, \varepsilon\big)\Big). \end{aligned} \tag{2.2}$$

Certainly the function $\varphi'(t, x)$ satisfies the variational equation

$$v' = g_y\big(x, \varphi(t, x)\big)v. \tag{2.3}$$

We also consider the dual variational system

$$w' = -g_y^*\big(x, \varphi(t, x)\big)w. \tag{2.4}$$

We suppose

(H2) There are smooth basis

$$\big\{v_0(t, x), v_1(t, x), \dots, v_r(t, x)\big\} \quad \text{and} \quad \big\{w_0(t, x), w_1(t, x), \dots, w_r(t, x)\big\}$$

of 1-periodic solutions of (2.3) and (2.4), respectively, for any $x \in \Omega$. We assume that $v_0(t, x) = \varphi'(t, x)$.

We consider the Banach spaces

$$\begin{aligned} X &= \big\{x \in C\big(\mathbb{R}, \mathbb{R}^n\big) \mid x(t+1) = x(t)\ \forall t \in \mathbb{R}\big\}, \\ Y &= \big\{y \in C\big(\mathbb{R}, \mathbb{R}^m\big) \mid y(t+1) = y(t)\ \forall t \in \mathbb{R}\big\} \end{aligned}$$

and then the projections

$$P_1 : X \to X, \qquad P_x : Y \to Y$$

defined by

$$\begin{aligned}
&P_1 x := x(t) - \int_0^1 x(s)\,ds,\\
&P_x y := y(t) - q_0 w_0(t,x) - q_1 w_1(t,x) - \cdots - q_r w_r(t,x),\\
&(q_0, q_1, \ldots, q_r)^* := A(x)^{-1}\left(\int_0^1 \bigl(y(t), w_0(t,x)\bigr)\,dt, \ldots, \int_0^1 \bigl(y(t), w_r(t,x)\bigr)\,dt\right)^*,
\end{aligned}$$

where $(\cdot,\cdot)$ is the scalar product on $\mathbb{R}^m$ and $A(x) : \mathbb{R}^r \to \mathbb{R}^r$ is the matrix given by

$$A(x) := \left(\int_0^1 \bigl(w_i(t,x), w_j(t,x)\bigr)\,dt\right)_{i,j=1}^r.$$

Now we take the changes

$$\begin{aligned}
&\varepsilon \leftrightarrow \varepsilon^2, \quad x = u + x_1, \quad u \in X, \quad P_1 u = u, \quad x_1 \in \mathbb{R}^n,\\
&y = v + \varepsilon \sum_{i=1}^r \beta_i v_i(t, u + x_1), \quad \int_0^1 \bigl(v(t), v_i(t)\bigr)\,dt = 0, \quad i = 0, 1, \ldots, r,
\end{aligned}$$

in (2.2), and also using projections P_1, P_x, we obtain

$$\begin{aligned}
&u' = \varepsilon^2 P_1 f\left(u + x_1, v + \varepsilon \sum_{i=1}^r \beta_i v_i(t, u + x_1) + \varphi(t, u + x_1), t + \alpha, \varepsilon^2\right),\\
&v' - g_y\bigl(x_1, \varphi(t, x_1)\bigr)v = P_{x_1} H(t),
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&\int_0^1 f\left(u + x_1, v + \varepsilon \sum_{i=1}^r \beta_i v_i(t, u + x_1) + \varphi(t, u + x_1), t + \alpha, \varepsilon^2\right) dt = 0,\\
&\frac{1}{\varepsilon^2}\int_0^1 \bigl(H(t), w_j(t, u + x_1)\bigr)\,dt = 0, \quad j = 0, 1, \ldots, r,
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
H(t) := {}& g\left(u + x_1, v + \varepsilon \sum_{i=1}^r \beta_i v_i(t, u + x_1) + \varphi(t, u + x_1)\right) - g\bigl(u + x_1, \varphi(t, u + x_1)\bigr)\\
&- g_y\bigl(u + x_1, \varphi(t, u + x_1)\bigr)\left(v + \varepsilon \sum_{i=1}^r \beta_i v_i(t, u + x_1)\right)
\end{aligned}$$

$$+\big(g_y\big(u+x_1,\varphi(t,u+x_1)\big)-g_y\big(x_1,\varphi(t,x_1)\big)\big)v-\varepsilon^3\sum_{i=1}^r\beta_i v_{ix}(t,u+x_1)$$

$$\times P_1 f\left(u+x_1,v+\varepsilon\sum_{i=1}^r\beta_i v_i(t,u+x_1)+\varphi(t,u+x_1),t+\alpha,\varepsilon^2\right)$$

$$+\varepsilon^2 h\left(u+x_1,v+\varepsilon\sum_{i=1}^r\beta_i v_i(t,u+x_1)+\varphi(t,u+x_1),t+\alpha,\varepsilon^2\right)$$

$$-\varepsilon^2\varphi_x(t,u+x_1)\times P_1 f\left(u+x_1,v+\varepsilon\sum_{i=1}^r\beta_i v_i(t,u+x_1)\right.$$

$$\left.+\varphi(t,u+x_1),t+\alpha,\varepsilon^2\right).$$

Moreover, we have

$$H(t)=\frac{1}{2}g_{yy}\big(u+x_1,\varphi(t,u+x_1)\big)\left(v+\varepsilon\sum_{i=1}^r\beta_i v_i(t,u+x_1)\right)^2$$

$$+\varepsilon^2 h\left(u+x_1,v+\varepsilon\sum_{i=1}^r\beta_i v_i(t,u+x_1)+\varphi(t,u+x_1),t+\alpha,\varepsilon^2\right)$$

$$-\varepsilon^2\varphi_x(t,u+x_1)\times P_1 f\left(u+x_1,v+\varepsilon\sum_{i=1}^r\beta_i v_i(t,u+x_1)\right.$$

$$\left.+\varphi(t,u+x_1),t+\alpha,\varepsilon^2\right)$$

$$+\mathrm{O}\left(\left(v+\varepsilon\sum_{i=1}^r\beta_i v_i(t,u+x_1)\right)^3\right)+\mathrm{O}(u)v+\mathrm{O}\big(\varepsilon^3\big).$$

Now, we can uniquely solve (2.5) by means of the implicit function theorem with $u=\mathrm{O}(\varepsilon^2)$ and $v=\mathrm{O}(\varepsilon^2)$. Then we insert these solutions to (2.6) to get the bifurcation equation

$$\begin{aligned}0=G(x_1,\beta,\alpha,\varepsilon)&=G_0(x_1,\beta,\alpha)+G_1(x_1,\beta,\alpha)\varepsilon+\cdots\\&\quad+G_p(x_1,\beta,\alpha)\varepsilon^p+Q_p(x_1,\beta,\alpha,\varepsilon)\varepsilon^{p+1}\\&:=q_p(x_1,\beta,\alpha,\varepsilon)+Q_p(x_1,\beta,\alpha,\varepsilon)\varepsilon^{p+1},\end{aligned}\tag{2.7}$$

where $\beta:=(\beta_1,\beta_2,\dots,\beta_r)$ and $G(x_1,\beta,\alpha,\varepsilon)$ is the left-hand side of (2.6). By using the Brouwer degree method like in [2], we obtain the following result.

Theorem 2.1. *If there is an open bounded subset $\mathcal{O}\subset\Omega\times\mathbb{R}^r\times\mathbb{R}$ and a constant $c_p>0$ such that*

$$\big|q_p(x_1,\beta,\alpha,\varepsilon)\big|\geqslant c_p\varepsilon^p$$

on the boundary $\partial\mathcal{O}$ for any $\varepsilon>0$ small, and $\deg(q_p(\cdot,\cdot,\cdot,\varepsilon),\mathcal{O},0)\neq 0$. *Then* (2.1) *has a* 1*-periodic solution for $\varepsilon>0$ small.*

Proof. We consider the homotopy

$$G(x_1,\beta,\alpha,\varepsilon,\lambda):=q_p(x_1,\beta,\alpha,\varepsilon)+\lambda Q_p(x_1,\beta,\alpha,\varepsilon)\varepsilon^{p+1}$$

for $\lambda\in[0,1]$. Then the assumptions of this theorem implies that $G(\cdot,\cdot,\cdot,\varepsilon,\lambda)\neq 0$ on $\partial\mathcal{O}$ for any $\varepsilon>0$ small and $\lambda\in[0,1]$. Hence [7]

$$\deg\bigl(G(\cdot,\cdot,\cdot,\varepsilon),\mathcal{O},0\bigr)=\deg\bigl(q_p(\cdot,\cdot,\cdot,\varepsilon),\mathcal{O},0\bigr)\neq 0.$$

So (2.7) is solvable for any $\varepsilon>0$ small. The proof is finished. □

We can immediately see from (2.5) and (2.6) that

$$\begin{aligned}G_0(x_1,\beta,\alpha)=\Biggl(&\int_0^1 f\bigl(x_1,\varphi(t,x_1),t+\alpha,0\bigr)\,dt,\\&\sum_{i,j=1}^r\beta_i\beta_j a_{ijk}(x_1)+\int_0^1\Bigl(h\bigl(x_1,\varphi(t,x_1),t+\alpha,0\bigr)\\&-\varphi_x(t,x_1)P_1 f\bigl(x_1,\varphi(t,x_1),t+\alpha,0\bigr),w_k(t,x_1)\Bigr)\,dt\Biggr),\end{aligned}\tag{2.8}$$

where $k=0,1,2,\dots,r$ and

$$a_{ijk}(x_1):=\frac{1}{2}\int_0^1\Bigl(g_{yy}\bigl(x_1,\varphi(t,x_1)\bigr)\bigl(v_i(t,x_1),v_j(t,x_1)\bigr),w_k(t,x_1)\Bigr)\,dt.$$

But since the zero points of (2.8) are the same as for the next function

$$\begin{aligned}Q_0(x_1,\beta,\alpha)=\Biggl(&\int_0^1 f\bigl(x_1,\varphi(t,x_1),t+\alpha,0\bigr)\,dt,\\&\sum_{i,j=1}^r\beta_i\beta_j a_{ijk}(x_1)+\int_0^1\Bigl(h\bigl(x_1,\varphi(t,x_1),t+\alpha,0\bigr)\\&-\varphi_x(t,x_1)f\bigl(x_1,\varphi(t,x_1),t+\alpha,0\bigr),w_k(t,x_1)\Bigr)\,dt\Biggr),\end{aligned}\tag{2.9}$$

where again $k=0,1,2,\dots,r$, we use (2.9) instead of (2.8). We also note that $G_0(x_1,\beta,\alpha)$ and $Q_0(x_1,\beta,\alpha)$ are connected by the homotopy

$$\begin{aligned}G_\lambda(x_1,\beta,\alpha)=\Biggl(&\int_0^1 f\bigl(x_1,\varphi(t,x_1),t+\alpha,0\bigr)\,dt,\\&\sum_{i,j=1}^r\beta_i\beta_j a_{ijk}(x_1)+\int_0^1\Bigl(h\bigl(x_1,\varphi(t,x_1),t+\alpha,0\bigr)\end{aligned}$$

$$- \varphi_x(t,x_1)\big((1-\lambda)P_1+\lambda\big)\mathbb{I}f\big(x_1,\varphi(t,x_1),t+\alpha,0\big),w_k(t,x_1)\big)\,dt\Bigg),$$

for $\lambda \in [0,1]$, and $G_\lambda(x_1,\beta,\alpha) \neq 0$ on $\partial\Omega$ if and only if $G_0(x_1,\beta,\alpha) \neq 0$ on $\partial\Omega$.

Of course, higher-order terms $G_i(x_1,\beta,\alpha)$, $i \geqslant 1$ are much more complicated, for this reason, we do not derive their general forms.

When $y' = g(x,y)$ has some symmetries then very often instead of condition (H1) the following condition holds

(C1) The equation $y' = g(x,y)$ has a smooth family $\varphi(t,x,\theta)$ of 1-periodic solution for any $x \in \Omega$ and $\theta \in \Gamma$, where $\Omega \subset \mathbb{R}^n$, $\Gamma \subset \mathbb{R}^r$ are open bounded subsets.

Then we can repeat the above procedure to (2.1) with the following modification: Eq. (2.3) has the form

$$v' = g_y\big(c,\varphi(t,x,\theta)\big)v. \tag{2.10}$$

We note that $\varphi'(t,x,\theta)$, $\varphi_{\theta_i}(t,x,\theta)$, $i=1,2,\ldots,r$, $\theta=(\theta_1,\theta_2,\ldots,\theta_r)$ are 1-periodic solutions of (2.10). We suppose

(C2) The family $\varphi(t,x,\theta)$ is nondegenerate, i.e. the functions $\widetilde{v}_0(t,x,\theta) := \varphi'(t,x,\theta)$, $\widetilde{v}_i(t,x,\theta) := \varphi_{\theta_i}(t,x,\theta)$, $i=1,2,\ldots,r$, form a basis of the space of 1-periodic solutions of (2.10).

Condition (C2) implies that there is a smooth basis $\widetilde{w}_j(t,x,\theta)$, $j=0,1,\ldots,r$, of the space of 1-periodic solutions of the adjoint system to (2.10) given by

$$w' = -g_y^*\big(c,\varphi(t,x,\theta)\big)w.$$

Now, we keep the projection P_1, but we change P_x to $P_{x,\theta}: Y \to Y$ defined by

$$P_{x,\theta}y := y(t) - \widetilde{q}_0\widetilde{w}_0(t,x,\theta) - \widetilde{q}_1\widetilde{w}_1(t,x,\theta) - \cdots - \widetilde{q}_r\widetilde{w}_r(t,x,\theta),$$

$$(\widetilde{q}_0,\widetilde{q}_1,\ldots,\widetilde{q}_r)^* := \widetilde{A}(x,\theta)^{-1}\left(\int_0^1\big(y(t),\widetilde{w}_0(t,x,\theta)\big)\,dt,\ldots,\int_0^1\big(y(t),\widetilde{w}_r(t,x,\theta)\big)\,dt\right)^*,$$

where $\widetilde{A}(x,\theta): \mathbb{R}^r \to \mathbb{R}^r$ is the matrix given by

$$\widetilde{A}(x,\theta) := \left(\int_0^1\big(\widetilde{w}_i(t,x,\theta),\widetilde{w}_j(t,x,\theta)\big)\,dt\right)_{i,j=1}^r.$$

Then in (2.1) we take the changes

$$x = u + x_1,\quad u \in X,\quad P_1u = u,\quad x_1 \in \mathbb{R}^n,$$

$$y = v + \varphi(t,u+x_1,\theta),\quad \int_0^1\big(v(t),\widetilde{v}_i(t)\big)\,dt = 0,\quad i=0,1,\ldots,r$$

and also using projections P_1, $P_{x,\theta}$, we obtain

$$\begin{aligned} &u' = \varepsilon P_1 f\big(u + x_1, v + \varphi(t, u + x_1, \theta), t + \alpha, \varepsilon\big), \\ &v' - g_y\big(x_1, \varphi(t, x_1, \theta)\big)v = P_{x_1,\theta} \widetilde{H}(t), \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\int_0^1 f\big(u + x_1, v + \varphi(t, u + x_1, \theta), t + \alpha, \varepsilon\big)\, dt = 0, \\ &\frac{1}{\varepsilon}\int_0^1 \big(\widetilde{H}(t), \widetilde{w}_j(t, u + x_1, \theta)\big)\, dt = 0, \quad j = 0, 1, \ldots, r, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \widetilde{H}(t) := {} & g\big(u + x_1, v + \varphi(t, u + x_1, \theta)\big) \\ & - g\big(u + x_1, \varphi(t, u + x_1, \theta)\big) - g_y\big(x_1, \varphi(t, u + x_1, \theta)\big)v \\ & + \varepsilon\big(h\big(u + x_1, v + \varphi(t, u + x_1, \theta), t + \alpha, \varepsilon\big) \\ & - \varphi_x(t, u + x_1, \theta) P_1 f\big(u + x_1, v + \varphi(t, u + x_1, \theta), t + \alpha, \varepsilon\big)\big). \end{aligned}$$

Now by using again the implicit function theorem, we can solve (2.11) to get $u = \mathrm{O}(\varepsilon)$ and $v = \mathrm{O}(\varepsilon)$. Then we insert these solutions to (2.12) to get the bifurcation equation

$$\begin{aligned} 0 = \widetilde{G}(x_1, \theta, \alpha, \varepsilon) &= \widetilde{G}_0(x_1, \theta, \alpha) + \widetilde{G}_1(x_1, \theta, \alpha)\varepsilon + \cdots \\ &\quad + \widetilde{G}_p(x_1, \theta, \alpha)\varepsilon^p + \widetilde{Q}_p(x_1, \theta, \alpha, \varepsilon)\varepsilon^{p+1} \\ &:= \widetilde{q}_p(x_1, \theta, \alpha, \varepsilon) + \widetilde{Q}_p(x_1, \theta, \alpha, \varepsilon)\varepsilon^{p+1}, \end{aligned}$$

where $\widetilde{G}(x_1, \beta, \alpha, \varepsilon)$ is the left-hand side of (2.12). So the Brouwer degree method again gives the following result.

Theorem 2.2. *If there is an open bounded subset $\mathcal{O} \subset \Omega \times \Gamma \times \mathbb{R}$ and a constant $\widetilde{c}_p > 0$ such that*

$$\big|\widetilde{q}_p(x_1, \beta, \alpha, \varepsilon)\big| \geqslant \widetilde{c}_p \varepsilon^p$$

on the boundary $\partial\mathcal{O}$ for any $\varepsilon > 0$ small, and $\deg(\widetilde{q}_p(\cdot, \cdot, \cdot, \varepsilon), \mathcal{O}, 0) \neq 0$. *Then* (2.1) *has a* 1*-periodic solution for $\varepsilon > 0$ small.*

We can immediately see from (2.11) and (2.12) that

$$\begin{aligned} \widetilde{G}_0(x_1, \theta, \alpha) = \bigg(&\int_0^1 f\big(x_1, \varphi(t, x_1, \theta), t + \alpha, 0\big)\, dt, \int_0^1 \big(h\big(x_1, \varphi(t, x_1, \theta), t + \alpha, 0\big) \\ & - \varphi_x(t, x_1, \theta) P_1 f\big(x_1, \varphi(t, x_1, \theta), t + \alpha, 0\big), \widetilde{w}_k(t, x_1, \theta)\big)\, dt \bigg) \end{aligned} \tag{2.13}$$

for $k = 0, 1, 2, \ldots, r$. Here we use again the function

$$\widetilde{Q}_0(x_1,\theta,\alpha)=\left(\int_0^1 f\big(x_1,\varphi(t,x_1,\theta),t+\alpha,0\big)\,dt,\ \int_0^1 \big(h\big(x_1,\varphi(t,x_1,\theta),t+\alpha,0\big)\right.$$
$$\left.-\ \varphi_x(t,x_1,\theta)f\big(x_1,\varphi(t,x_1,\theta),t+\alpha,0\big),\widetilde{w}_k(t,x_1,\theta)\big)\,dt\right) \tag{2.14}$$

for $k=0,1,2,\ldots,r$, instead of (2.13).

Again, the higher-order terms $\widetilde{G}_i(x_1,\theta,\alpha)$, $i\geqslant 1$, are still rather complicated, so we do not derive them.

3. Examples

We present in this section two examples, where we apply Theorems 2.1 and 2.2.

Example 1. We first consider the system

$$\begin{aligned}
y_1' &= \big(x^2+1\big)\big(y_1^2+y_2^2\big)y_2+\varepsilon y_1,\\
y_2' &= -\big(x^2+1\big)\big(y_1^2+y_2^2\big)y_1-\varepsilon y_2^3,\\
x' &= \varepsilon\big(y_1^3\sin 2\pi t+y_2\cos 2\pi t\big).
\end{aligned} \tag{3.1}$$

The unperturbed system

$$\begin{aligned}
y_1' &= \big(x^2+1\big)\big(y_1^2+y_2^2\big)y_2,\\
y_2' &= -\big(x^2+1\big)\big(y_1^2+y_2^2\big)y_1
\end{aligned} \tag{3.2}$$

has a smooth family of 1-periodic solutions

$$\varphi(t,x)=\sqrt{\frac{2\pi k}{x^2+1}}(\sin 2\pi kt,\cos 2\pi kt) \tag{3.3}$$

for $k\in\mathbb{Z}\setminus\{0\}$. The linearization of (3.2) along (3.3) is

$$\begin{aligned}
v_1' &= 2\pi k\big(\sin 4\pi kt\,v_1+(2+\cos 4\pi kt)v_2\big),\\
v_2' &= -2\pi k\big((2-\cos 4\pi kt)v_1+\sin 4\pi kt\,v_2\big)
\end{aligned} \tag{3.4}$$

and the adjoint system is

$$\begin{aligned}
w_1' &= 2\pi k\big(-\sin 4\pi kt\,w_1+(2-\cos 4\pi kt)w_2\big),\\
w_2' &= 2\pi k\big(-(2+\cos 4\pi kt)w_1+\sin 4\pi kt\,w_2\big).
\end{aligned} \tag{3.5}$$

Now (3.4) has the solutions

$$\begin{aligned}
v_0(t,x) &= (\cos 2\pi kt,-\sin 2\pi kt),\\
\widetilde{v}(x,t) &= (\sin 2\pi kt+4\pi kt\cos 2\pi kt,\cos 2\pi kt-4\pi kt\sin 2\pi kt).
\end{aligned}$$

Hence $v_0(x,t)$ is a basis of 1-periodic solutions of (3.4). Furthermore, the function

$$w_0(t,x)=(\sin 2\pi kt,\cos 2\pi kt)$$

is a basis of 1-periodic solutions of (3.5). We note that now we do not have parameters β. For simplicity we take $k = 1$. After some computations, the function Q_0 of (2.9) for this case (3.1) has the form

$$\begin{aligned} Q_0(x_1, \alpha) = \frac{\sqrt{\pi}}{2\sqrt{2}(x_1^2+1)^{3/2}} \times \bigg(&\big(2 + 3\pi + 2x_1^2\big)\cos 2\alpha\pi, \\ &2 - 3\pi + 2x_1^2 + \frac{x\sqrt{2\pi}}{(x_1^2+1)^{3/2}}\big(2 + 3\pi + 2x_1^2\big)\cos 2\alpha\pi \bigg). \end{aligned} \tag{3.6}$$

We immediately see that (3.6) has a simple root

$$x_1 = \sqrt{\frac{3\pi - 2}{2}}, \qquad \alpha = \frac{1}{4}.$$

Theorem 2.1 gives the existence of 1-periodic solution of (3.1) for any $\varepsilon \neq 0$ small. This 1-periodic solution of (3.1) is $\mathrm{O}(\varepsilon)$-near to

$$\left(-\frac{2\sqrt{3}}{3}\cos 2\pi t, \frac{2\sqrt{3}}{3}\sin 2\pi t, \sqrt{\frac{3\pi - 2}{2}}\right).$$

Example 2. Finally, we consider the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1 - \big(x^2+1\big)\big(y_1^2+y_3^2\big)y_1 - \varepsilon\delta y_2 - \varepsilon\mu_1(y_1 - y_3) - \varepsilon\mu_2\cos 2\pi t, \\ y_3' &= y_4, \\ y_4' &= -y_3 - \big(x^2+1\big)\big(y_1^2+y_3^2\big)y_3 - \varepsilon\delta y_3 - \varepsilon\mu_1(y_3 - y_1) - \varepsilon\mu_2\cos 2\pi t, \\ x' &= \varepsilon(y_1 \sin 2\pi t + y_3\cos 2\pi t), \end{aligned} \tag{3.7}$$

where δ, μ_1, μ_2 are positive parameters. The unperturbed system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1 - \big(x^2+1\big)\big(y_1^2+y_3^2\big)y_1, \\ y_3' &= y_4, \\ y_4' &= -y_3 - \big(x^2+1\big)\big(y_1^2+y_3^2\big)y_3 \end{aligned} \tag{3.8}$$

has a smooth family of periodic solutions

$$y(t, x, \theta, k) = \big(\cos\theta v(t, x, k), \cos\theta v(t, x, k)', \sin\theta v(t, x, k), \sin\theta v(t, x, k)'\big), \tag{3.9}$$

where

$$v(t, x, k) = \frac{\sqrt{2}k}{\sqrt{(1 - 2k^2)(x^2+1)}}\,\mathrm{cn}\,\frac{t}{\sqrt{1 - 2k^2}}$$

and cn is the Jacobi elliptic function [4] and k is the elliptic modulus. The function $y(t, x, \theta, k)$ has the period $T(k) = 4K(k)\sqrt{1 - 2k^2}$ for the complete elliptic integral $K(k)$ of the first kind. We note $T(0) = 2\pi$ and $T(\sqrt{2}/2) = 0$. By numerically solving the equation $T(k) = 1$, we find its unique solution $k_0 \doteq 0.700595$. We also see from the graph of function $T(k)'$ that $T(k_0)' \neq 0$. So we fix $k = k_0$ and take

$$\varphi(t,x,\theta)=y(t,x,\theta,k_0).$$

We also note that (3.8) has the form

$$\ddot{w}+\big(1+\big(x^2+1\big)\|w\|^2\big)w=0 \tag{3.10}$$

for $w=(y_1,y_3)$. By putting

$$\Gamma(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

wee see that if $w(t)$ solves (3.10) then also $\Gamma(\theta)w(t)$ solves it. By taking $w(t)=(v(t,x,k),0)$, we get (3.9). The linearization of (3.10) at w has the form

$$\ddot{z}+\big(1+\big(x^2+1\big)\|w\|^2\big)z+2\big(x^2+1\big)\langle w,z\rangle w=0, \tag{3.11}$$

where $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbb{R}^2$ and $\|\cdot\|$ is the corresponding norm. Furthermore, we can easily check that if $z=\Gamma(\theta)z_1$ and $w=\Gamma(\theta)w_1$, then

$$\ddot{z}_1+\big(1+\big(x^2+1\big)\|w_1\|^2\big)z_1+2\big(x^2+1\big)\langle w_1,z_1\rangle w=0. \tag{3.12}$$

Consequently, in order to study the linearization of (3.8) (or (3.11)) at $\varphi(t,x,\theta)$, we study the linearization of (3.12) at $w_1(t)=(v(t,x),0)$, $v(t,x)=v(t,x,k_0)$ which has the form

$$\begin{aligned}&v_1'=v_2,\\ &v_2'=-\big(1+3w(t,k_0)\big)v_1,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&v_3'=v_4,\\ &v_4'=-\big(1+w(t,k_0)\big)v_3\end{aligned} \tag{3.14}$$

for

$$w(t,k)=\frac{2k^2}{1-2k^2}\,\mathrm{cn}^2\,\frac{t}{\sqrt{1-2k^2}}.$$

Eq. (3.13) has an 1-periodic solution $v(t,x)'$ and a non-1-periodic solution $\frac{\partial}{\partial k}v(t,x,k_0)$. Eq. (3.14) has a 1-periodic solution $v(t,x)$ and by solving numerically (3.14) with initial value conditions $v_3(0)=0$, $v_4(0)=1$, we see that the second solution of (3.14) is non-1-periodic. Consequently, condition (C2) is satisfied. Moreover, we have

$$\begin{aligned}
&\widetilde{v}_0(t,x,\theta)=\big(\cos\theta v(t,x)',\cos\theta v(t,x)'',\sin\theta v(t,x)',\sin\theta v(t,x)''\big),\\
&\widetilde{v}_1(t,x,\theta)=\big(-\sin\theta v(t,x),-\sin\theta v(t,x)',\cos\theta v(t,x),\cos\theta v(t,x)'\big),\\
&\widetilde{w}_0(t,x,\theta)=\big(-\cos\theta v(t,x)'',\cos\theta v(t,x)',-\sin\theta v(t,x)'',\sin\theta v(t,x)'\big),\\
&\widetilde{w}_1(t,x,\theta)=\big(\sin\theta v(t,x)',-\sin\theta v(t,x),-\cos\theta v(t,x)',\cos\theta v(t,x)\big).
\end{aligned}$$

Now we insert the above formulas to (2.14) and by using the evenness of function cn, after some computations we get the first-order bifurcation function

$$\widetilde{Q}_0(x,\theta,\alpha)=\big(\widetilde{Q}_{01}(x,\theta,\alpha),\widetilde{Q}_{02}(x,\theta,\alpha),\widetilde{Q}_{03}(x,\theta,\alpha)\big),$$

where

$$\widetilde{Q}_{01}(x,\theta,\alpha)=\frac{\sin(2\pi\alpha+\theta)}{\sqrt{x^2+1}}\int_0^1 w(t)\cos 2\pi t\,dt,$$

$$\widetilde{Q}_{02}(x,\theta,\alpha)=\frac{1}{(x^2+1)}\int_0^1\Big\{-\delta\dot{w}(t)^2+\mu_2(\cos\theta+\sin\theta)\sqrt{x^2+1}\sin 2\pi t\sin 2\pi\alpha\,\dot{w}(t)$$
$$+\frac{x}{(x^2+1)^{3/2}}\big(\dot{w}(t)^2-\ddot{w}(t)w(t)\big)w(t)\cos 2\pi t\sin(2\pi\alpha+\theta)\Big\}\,dt,$$

$$\widetilde{Q}_{0,3}(x,\theta,\alpha)=\frac{1}{x^2+1}\int_0^1\big\{\mu_1\cos 2\theta\,w(t)^2$$
$$-\sqrt{x^2+1}\mu_2(\cos\theta-\sin\theta)\cos 2\pi t\cos 2\pi\alpha\,w(t)\big\}\,dt,$$

and

$$w(t)=\frac{\sqrt{2}k_0}{\sqrt{1-2k_0^2}}\operatorname{cn}\frac{t}{\sqrt{1-2k_0^2}}.$$

Since $\int_0^1 w(t)\cos 2\pi t\,dt\sim 3.49859$, we see that $\widetilde{Q}_{01}(x,\theta,\alpha)=0$ gives two possibilities: either $2\pi\alpha+\theta=0$ or $2\pi\alpha+\theta=\pi$. Then $\widetilde{Q}_{02}(x,\theta,\alpha)=0$ implies

$$\sqrt{x^2+1}=\frac{\delta\int_0^1\dot{w}(t)^2\,dt}{\mu_2(\cos\theta+\sin\theta)\sin 2\pi\alpha\int_0^1\dot{w}(t)\sin 2\pi t\,dt}>1. \tag{3.15}$$

By inserting (3.15) into $\widetilde{Q}_{03}(x,\theta,\alpha)=0$ we get the following equivalent equation

$$\delta(\cos\theta-\sin\theta)\cos 2\pi\alpha\int_0^1\dot{w}(t)^2\,dt\int_0^1 w(t)\cos 2\pi t\,dt$$
$$-\mu_1\cos 2\theta(\cos\theta+\sin\theta)\sin 2\pi\alpha\int_0^1 w(t)^2\,dt\int_0^1\dot{w}(t)\sin 2\pi t\,dt=0. \tag{3.16}$$

First we consider the case $2\pi\alpha=-\theta$. From (3.15) we obtain

$$\frac{\delta}{\mu_2(\cos\theta+\sin\theta)\sin\theta}>0.02236, \tag{3.17}$$

while (3.16) gives

$$(\cos\theta-\sin\theta)\big\{6.38018\delta\cos\theta-\mu_1(\sin\theta+\cos\theta)^2\sin\theta\big\}=0. \tag{3.18}$$

We first consider in (3.18) the cases when either $\theta_0=\pi/4$ or $\theta_0=3\pi/4$, which are simple roots of (3.18) if

$$\delta/\mu_1\neq 0.313471. \tag{3.19}$$

Then by inserting $\theta=\theta_0$ into (3.17), we get

$$\delta/\mu_2 > 0.02236. \tag{3.20}$$

Now we consider that $\theta \neq \theta_0$, then (3.17) and (3.18) are equivalent to

$$A = 6.380018\delta/\mu_1 = \frac{(1+\tan\theta)^2}{1+\tan^2\theta}\tan\theta = \Psi_1(\tan\theta) \tag{3.21}$$

and

$$44.7227\delta/\mu_2 > \frac{1+\tan\theta}{1+\tan^2\theta}\tan\theta = \Psi_2(\tan\theta) > 0. \tag{3.22}$$

So we take $\theta \in (0, \pi/2)$. Then we can check that (3.21) is uniquely solvable in $\tan\theta$ as a function of $A \geqslant 0$, and inserting this solution into the right hand side of (3.22), we obtain the following condition

$$F(6.380018\delta/\mu_1) < 44.7227\delta/\mu_2, \tag{3.23}$$

where

$$F(A) = 3A\left(1 + A + \frac{A^2 - 4A + 1}{C^{1/3}} + C^{1/3}\right)^{-1},$$
$$C = 1 + 21A - 6A^2 + A^3 + 3\sqrt{6A + 42A^2 - 18A^3 + 3A^4}.$$

Finally, we consider the case $2\pi\alpha = \pi - \theta$. Then (3.21) and (3.22) are changing to

$$-6.380018\delta/\mu_1 = \Psi_1(\tan\theta). \tag{3.24}$$

and

$$-44.7227\delta/\mu_2 < \Psi_2(\tan\theta) < 0, \tag{3.25}$$

respectively. So we take $\theta \in (-\pi/4, 0)$. Now the situation is different: functions $\Psi_{1,2}$ are not invertible on interval $\mathcal{I} = [-1, 0]$. They are both non-positive on $\mathcal{I}$. Function Ψ_1 has the minimum -0.134884 on $\mathcal{I}$ at -0.295598, while function Ψ_2 has the minimum -0.207107 on $\mathcal{I}$ at -0.414214. So in order to solve (3.24) we suppose $-6.380018\delta/\mu_1 > -0.134884$, while the condition $-44.7227\delta/\mu_2 < -0.207107$ is sufficient for holding (3.25). We can put these two inequalities in the following one

$$0.00463021\mu_2 < \delta < 0.0211411\mu_1. \tag{3.26}$$

Summarizing, we arrive at the following result.

Theorem 3.1. *Under either conditions* (3.19) *and* (3.20)*, or* (3.23)*, or* (3.26)*, system* (3.7) *possesses an* 1*-periodic solution for any* $\varepsilon > 0$ *small.*

We note that if several conditions of Theorem 3.1 hold simultaneously, then we get multiple 1-periodic solutions.

When $\delta = 0$ then we get a different situation. The bifurcation function $\widetilde{Q}_0(x, \theta, \alpha)$ remains. Also we get that either $2\pi\alpha = -\theta$ or $2\pi\alpha = \pi - \theta$. Equations $\widetilde{Q}_{0j}(x, \theta, \alpha) = 0$, $j = 2, 3$ imply the following ones

$$(\cos\theta + \sin\theta)\sin\theta = 0,$$
$$(\cos\theta - \cos\theta)\big(7.0107\mu_1(\cos\theta + \sin\theta) - \mu_2\cos 2\pi\alpha\sqrt{x^2+1}\big) = 0. \tag{3.27}$$

By analyzing system (3.27), we get a solution:

$$\theta = 0, \quad \alpha = 0, \quad 7.0107\mu_1/\mu_2 = \sqrt{x^2+1}.$$

Consequently, we derive the following result.

Theorem 3.2. *If* $\delta = 0$ *and*

$$\mu_1 > 0.14264\mu_2,$$

then system (3.7) *possesses an* 1*-periodic solution for any* $\varepsilon > 0$ *small.*

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